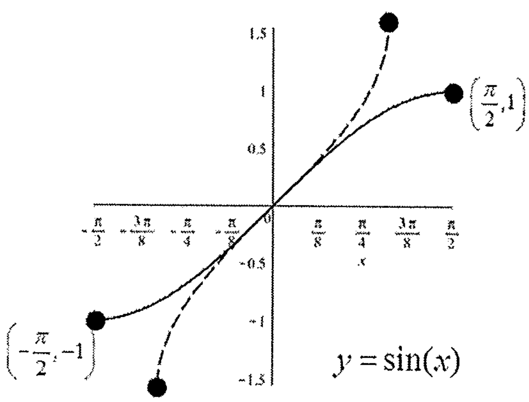
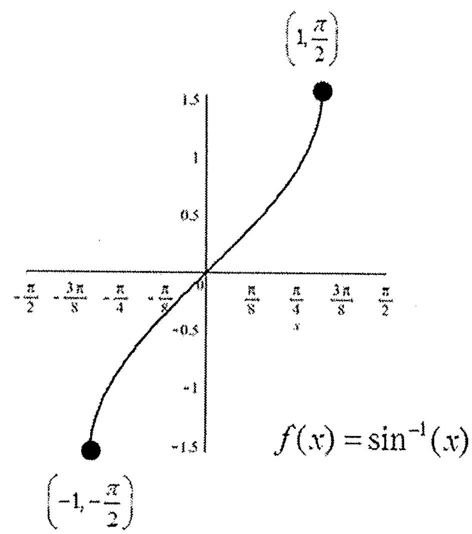
Inverse Trig Functions
and
Their Derivatives

$f(x) = \arcsin(x) = \sin^{-1}(x) =$ "the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x."

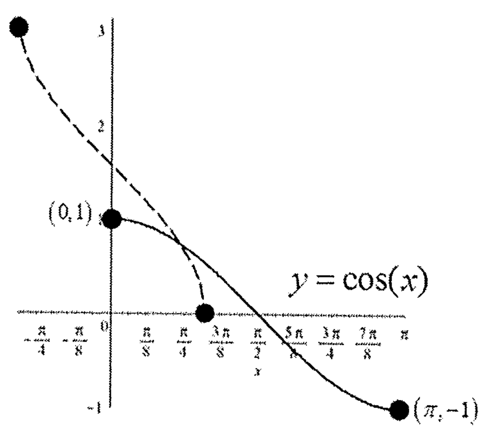
Domain: $[-1,1]$ Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

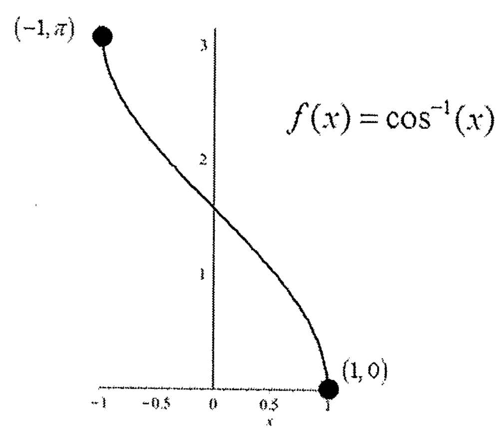




$f(x) = \arccos(x) = \cos^{-1}(x) =$ "the angle between 0 and π whose cosine is x."

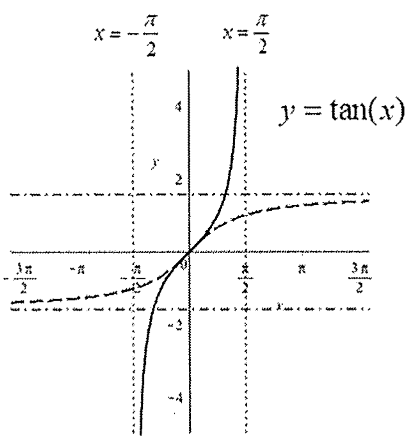
Domain: $[-1,1]$ Range: $[0,\pi]$

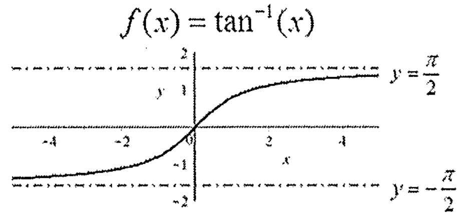




$f(x) = \arctan(x) = \tan^{-1}(x) =$ "the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tan is x."

Domain: $[-\infty, \infty]$ Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$





Derivatives of the Inverse Trig Functions

When we talked about the derivative of the logarithm, we argued that if f and f^{-1} are inverse functions, then it ought to be the case that

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}$$

We are now in a position to establish this more rigorously. All pairs of inverse function have the following property:

$$f \circ f^{1}(x) = f(f^{-1}(x)) = x \text{ for all } x.$$

Since these two functions are equal for all values of x, their derivatives are also equal.

$$\frac{d}{dx}(f(f^{-1}(x)) = \frac{d}{dx}x.$$

The chain rule gives us $f'\left(f^{-1}(x)\right)(f^{-1})'(x) = 1$. So

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))} .$$

So we can use this formula to deduce the derivatives of the arcsine, the arccosine and the arctangent:

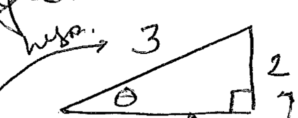
$$\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))}$$

$$\frac{d}{dx}\arccos(x) = \frac{1}{-\sin(\arccos(x))}$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{\sec^2(\arctan(x))}$$

The Triangle Trick: As it turns out, we can make these much more user-friendly by using a cool trick. When you have an inverse trig function on the inside and a trig function on the outside, you can rewrite the expression in a much more usable form. Consider the following example:

Calculate $\tan(\sin^{-1}(2/3))$. (0) First note that $\theta = \sin^{-1}(2/3)$ is an angle. (1) We can place it in side a right triangle:



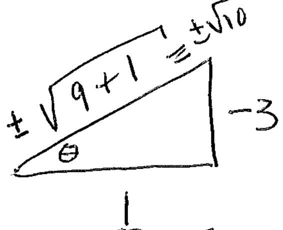
(2) We also know $\sin(\theta) = 2/3$

(3) By the pythagorean theorem $\sqrt{3^2 - 2^2} = \sqrt{5}$

So $\tan(\sin^{-1}(2/3)) = \tan(\theta) = \frac{2}{\sqrt{5}}$

Caution: this trick assumes that all angles are acute. (It assumes they will fit into a right triangle.) Sometimes we have to worry about the *sign* of the result. For example: $\cos(\tan^{-1}(-3))$

(0) $\theta = \tan^{-1}(-3)$ is still an angle. So we still put it into a Δ. We just keep in mind that sides can now have negative length.



(1) $\tan\theta = -3$ so -3 = opposite side; 1 = adj. side

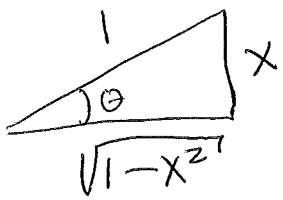
(2) Use Pythagorean thm. Hyp $= \pm\sqrt{10}$

(3) Because $\tan\theta$ is negative, $-\pi/2 \le \theta < 0$.

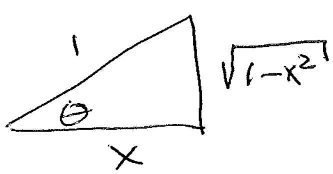
(4) The cosine of an angle in this quadrant is positive. $\cos(\tan^{-1}(-3)) = \cos(\theta) = \frac{3}{\sqrt{10}}$

Now we can simplify our formulas for the derivatives of the inverse trig functions:

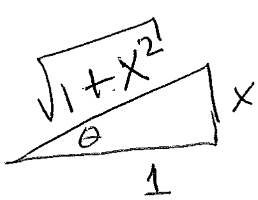
$$\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{1-x^2}}$$



$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sin(\arccos(x))} = -\frac{1}{\sin(\theta)} = -\frac{1}{\sqrt{1-x^2}}$$



$$\frac{d}{dx}\arctan(x) = \frac{1}{\sec^2(\arctan(x))} = \frac{1}{\sec^2(\theta)} = \frac{1}{\left(\sqrt{1+x^2}\right)^2} = \frac{1}{1+x^2}$$



So we have some new derivatives:

$f(x)$	$f'(x)$
$\arcsin(x) = \sin^{-1}(x)$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos(x) = \cos^{-1}(x)$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\arctan(x) = \tan^{-1}(x)$	$\dfrac{1}{1+x^2}$